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*On the General Properties of Algebraic Surfaces, by the Rev.
Dionysius Lardner, M. R. I. A.*

Read May 24, 1824.

(1) **SURFACES** represented by algebraic equations have many properties analogous to those of algebraic curves, which have been investigated in the twenty-first section of the first Part of my Geometry. These properties are, for the most part, investigated on principles exactly similar to those used in that section, and it will be therefore unnecessary to enter with the same minuteness into the details of the reasoning.

Surfaces, like plane curves, are classed according to the degrees of the equations which represent them, and for the same reasons. An equation between three variables may not always represent a surface of that degree marked by its dimensions. This happens when it is capable of being resolved into rational factors of inferior degrees, in which case it represents as many different surfaces of inferior orders. It may even represent only a system of planes, which takes place when it can be resolved into rational and real factors of the first degree. Thus the observations in (598) and (599) of the Geometry apply equally to surfaces represented by algebraic equations.

PROP.

(2) *To determine the number of given points through which an algebraic surface may be subjected to pass.*

If the coordinates of the points be successively substituted in the equation of the surface, there should be as many equa-

tions as there are constant quantities. The number of points therefore is equal to that of the constants of the equation. See (600) Geometry.

PROP.

(3) *To determine the greatest number of points in which a right line can meet an algebraic surface.*

Let x and y be eliminated by the equation of the surface and those of the right line, by substituting their values derived from the latter in the former. The result will be an equation of the form.

$$Ax^n + Bx^{n-1} + Cx^{n-2} \dots Mx + N = 0,$$

the surface being supposed of the n^{th} order. The number of real roots of this equation cannot exceed n , and it therefore follows that the number of points where the right line meets the surface cannot exceed n . As the roots may some or all be impossible, the line may meet the surface in a less number of points, or in none.

Hence *every algebraic surface may be intersected by a right line in as many points as there are units in the exponent of its order, but not in more.*

PROP.

(4) *To determine the curve of the highest order in which two algebraic surfaces, the exponents of whose orders are m and n can intersect.*

Assuming the general equations of the m^{th} and n^{th} orders between three variables, if the variables were successively eliminated the

results would be three equations between each pair of the variables, and these equations would be of the mn^{th} degree. (See note on (602,) Geometry.) Hence the curve sought is of the mn^{th} degree.

PROP.

(5) *Three right lines intersecting at the same point, intersect an algebraic surface; to determine the relation between the continued product of the intercepts of each between their common intersection and the surface.*

Let these right lines themselves be assumed as axes of coordinates. In order to determine their intercepts between the origin and the surface let each pair of variables be successively suppose = 0. The result will be three equations of the forms

$$\begin{array}{llll} Ax^n + Bx^{n-1} + Cx^{n-2} & . & . & . & Mx + N = 0. \\ A'y^n + B'y^{n-1} + C'y^{n-2} & . & . & . & M'y + N = 0. \\ A''z^n + B''z^{n-1} + C''z^{n-2} & . & . & . & M''z + N = 0. \end{array}$$

The continued products of the roots of these equations are respectively $\frac{N}{A}$, $\frac{N}{A'}$, $\frac{N}{A''}$, hence the ratio of each pair of these products is the reciprocal ratio of the coefficients of the highest powers of the variables respectively which enter the equations whose roots are their factors. As the transformation of origin does not affect the values of A, A', A'' , these ratios remain the same for all parallel systems of axes. Therefore *if three right lines parallel to three right lines given in position and intersecting at the same point intersect an algebraic surface of the n^{th} order, the continued product*

of their segments between their common intersection and the surface will be in a constant ratio. (Geometry, 606.)

(6) If a general equation of the n^{th} degree between three variables be arranged according to the dimensions of one of them, z ,

$$C. z^n + A_1 z^{n-1} + A_2 z^{n-2} + A_3 z^{n-3} + \dots + A_n = 0.$$

it is evident that A_1 will be a formula of the first degree between x and y , A_2 one of the second degree between these variables, and in general A_n a formula of the n^{th} degree between them.

For each system of values of x and y this equation gives n values of z , which when real determine as many points of the surface. The sum of these values is $-\frac{A_1}{C}$. If a surface be assumed, represented by the equation

$$z = -\frac{A_1}{nC}$$

it will have the property of intersecting the ordinate z , so that the sum of the intercepts between it and the surface on the one side shall be equal to the sum of the intercepts on the other side. Since A_1 has the form $Ay + Bx + D$, the equation of this surface is

$$Ay + Bx + nc z + D = 0;$$

it is therefore a plane. This plane therefore intersects a system of parallels to the axis of z , so that the sums of the intercepts between it and the surface on each side are equal. Such a plane may be called a diametral plane from its analogy to the diameter of a plane curve.

Such planes may be found relatively to every system of axes of coordinates. It therefore follows that an algebraic surface has an infinite number of diametral planes.

The sum of every two values of z is $\frac{\Lambda_2}{c}$ and the number of these products is $\frac{n \cdot n-1}{1 \cdot 2}$. Hence a surface represented by the equation

$$z^2 - \frac{\Lambda_1}{n c} z + \frac{2 \Lambda_2}{n \cdot n-1 \cdot c} = 0.$$

will have the same diametral plane with the given surface, and also the rectangle under the coincident values of z , for this surface will be equal to the $\frac{n \cdot n-1}{1 \cdot 2}$ th part of the sum of rectangles under every pair of corresponding values of z for the proposed surface. It follows therefore that the sum of the positive rectangles under the intercepts of the values of z between this surface and the given surface is equal to the sum of the negative rectangles. Since Λ_2 is a formula of the second degree between x and y , it follows that the surface possessing this property is one of the second degree.

It follows in general that a series of surfaces may be thus determined of degrees inferior to that of the given surface, and having properties with respect to it similar to those of curvilinear diameters with respect to plane curves. Such surfaces may be called *diametral surfaces*. (Geometry, 607.)

An *absolute diametral plane* is one which bisects its ordinates. It is evident all diametral planes of surfaces of the second degree must be absolute.

In order that a surface should admit an absolute diametral

plane, it is necessary that a transformation of coordinates, which would make all the terms involving odd powers of the variables disappear, should be possible ; and as this is not always the case, surfaces of degrees exceeding the second may not have an absolute diametral plane.

(7) The centre of a surface is a point which bisects all right lines passing through it and terminated in the surface.

PROP.

(8) *To determine when an algebraic surface has a centre.*

If the origin be at the centre, the value of z corresponding to $+x$ and $+y$ ought to be equal and opposite to that corresponding to $-x$ and $-y$; or which is the same, it is necessary that the equation should not be affected by changing the signs of the three coordinates. In order that this should take place, it is necessary that the sum of the exponents of the variables should be either *even* or *odd* in all the terms of the equation. If a change of origin which will produce this effect be possible, the surface has a centre ; otherwise not.